Effect of coupling and linear transformation of waves in shear flows

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A new linear mechanism of reciprocal transformation of waves and a corresponding energy transfer in shear flows is discovered. The effect is demonstrated on the simplest example — the two-dimensional waves in unbounded, parallel hydromagnetic flow with uniform velocity shear. The phenomenon discovered is of importance for magnetohydrodynamics and fusion plasma devices and for various terrestrial and astrophysical shear flows. Grasping the result became possible thanks to the nonmodal analyses of the perturbation evolution in the flow. [S1063-651X(96)11605-6]

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Oscillatory systems are prevalent in the nature and play an important role in the wide variety of physical processes. It should be emphasized that: (i) these kinds of dynamical systems usually consist of coupled oscillators and have several degrees of freedom; (ii) parameters of the oscillators in most cases are not constant but vary slowly in time [1]. The processes taking place in such systems are immensely complex and their study is applicable not only to the various branches of physics, but also to chemistry, biology, sociology, and other sciences. Investigation of these two key aspects of dynamics, closely related to each other in some physical situations, has become one of the most important interdisciplinary problems in applied mathematics [1–3]. Hydrodynamical and plasma flows constitute such oscillatory systems, where coupling, as a rule, is associated with nonlinear processes — wave decay processes [4], which ensures the mutual transformation of different wave modes. In plasma physics, a linear coupling phenomenon is also known: mutual transformation of different kinds of plasma waves arising due to a *spatial inhomogeneity* of a medium. For example, existence of the density inhomogeneity induces coupling between magnetohydrodynamic (MHD) oscillations [5] or the transformation of compressional-type waves into electromagnetic-type waves propagating across a density discontinuity [6], etc.

In the present paper, we describe a mechanism of the linear reciprocal transformation of wave modes arising in flows due to a *velocity inhomogeneity*. The effect is demonstrated for the simplest example: MHD waves in two-dimensional (2D), compressible, magnetized, unbounded parallel flow with uniform velocity shear (plane, magnetized Couette flow). The result is obtained by means of the nonmodal approach applied to the study of the evolution of small-scale perturbations in the flow. This effect makes more diverse the variety of processes taking place in shear flows. Under a traditional modal analysis, this phenomenon was not noticed for the following reason: the linear operators arising in shear flows (as becomes apparent in a number of recent contributions [7–13]) are non-normal. It results in a set of nonorthogonal eigenfunctions strongly interfering with each other. That is why analysis of distinct eigenfunction evolution, performed in the framework of the modal approach (without adequate consideration of the interference), is misleading in many respects.

In the recent past, the nonmodal approach (''Kelvin formalism'' [14]) came to be extensively used [7–13, 15–18] in the study of evolution of the perturbations in the shear flows. In this formalism, one considers the temporal evolution of *spatial Fourier harmonics* (''Kelvin modes'' [15]) of the perturbations without any spectral expansion in time. The wave number of each spatial Fourier harmonic (SFH) varies in time along the flow shear: in the linear approximation there exists a ''linear drift'' of SFH in the plane of wave numbers (**k** space) [11–13]. Owing to the advantages of the method, there were found transient growth of vortical perturbations in smooth hydrodynamic shear flows [7–10, 13–15]; anomalous growth of the slow magnetosonic waves in the 2D incompressible magnetized plane Couette flows [11]; a mechanism of the shear energy transfer to the acoustic perturbations in 2D, compressible plane Couette flows [12]. Besides, based on the nonmodal approach, results of recent investigations (see Refs. [7–10, 13]) reveal the potential for an adequate description of the transition to turbulence in smooth shear flows.

It is worthwhile to note that the importance of the nonmodal solutions of the MHD equations governing the Tokamak devices was also revealed in previous contributions [19]; in particular, the author considered the ballooning instabilities in Tokamak devices with sheared toroidal flow. Based on the covering space concept [20], which, in fact does not constrain any unstable solution to evolve as $\exp(i\omega t)$, it was shown that an adequate description of the ballooning mode can be performed considering the nonmodal solutions of the equations. Basically, the ballooning-mode eikonal representation used by the author is similar to the Kelvin formalism, but its mathematical formulation is different, since it is applied to the physical system with axial symmetry.

To attain the aim of this paper — to demonstrate a specific mechanism of linear transformation of the waves in

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flows with inhomogeneous mean velocity, let us consider the evolution of 2D waves in compressible, unbounded parallel flow with uniform velocity shear contained in an external regular magnetic field $\mathbf{B_0} \| \mathbf{U_0}$. It is known that in such cases, the mean velocity field is given by the formula $\mathbf{U_0}(Ay;\mathbf{0};\mathbf{0})$ (the X axis is directed along the regular velocity vector, the Y axis is directed along the shear), where the constant A (without loss of generality, we adopt $A>0$) parametrizes inhomogeneity of the background flow.

The simplicity of this model flow guarantees its applicability to a wide variety of terrestrial and astrophysical shear flows. The reason is simple: if one considers small-scale perturbations, an arbitrary smooth ''shear profile'' is, of course, approximately linear on the scales, much smaller than the length scale of the flow itself. A Goldreich–Lynden-Bell model [21], widely used since the 1960s for astrophysical shear flows, is a good example of such an approximation. Hence, the results of the forthcoming consideration should be relevant for various laboratory and space plasma shear flows.

The basic system of linearized equations governing the evolution of the small-scale, 2D perturbations in this flow is

$$(\partial_t + Ay\partial_x)d + \partial_x u_x + \partial_y u_y = 0, \tag{1}$$

$$(\partial_t + Ay\partial_x)u_x + Au_y = -c_s^2 \partial_x d, \tag{2}$$

$$(\partial_t + Ay\partial_x)u_y = -c_s^2 \partial_y d + c_A^2[\partial_x b_y - \partial_y b_x], \tag{3}$$

$$(\partial_t + Ay\partial_x)b_y = \partial_x u_y, \tag{4}$$

$$\partial_x b_x + \partial_y b_y = 0, \tag{5}$$

where $d \equiv \rho'/\rho_0$, $\mathbf{b} \equiv \mathbf{B}'/|\mathbf{B_0}|$, and c_s and c_A are the speed of sound and an Alfven velocity, respectively. Note that in Eqs. (2) and (3) we have used the polytropic equation of state $p = K\rho^\gamma$ to express the pressure perturbation by means of the density perturbation: $\mathrm{grad}(p') = (\partial p/\partial\rho)\mathrm{grad}(\rho') \equiv c_s^2 \mathrm{grad}(\rho')$. The starting point for the nonmodal analysis is to make the following substitution of variables [11–13], $x_1 = x - Ayt$; $y_1 = y$; $t_1 = t$, and rewrite the Eqs. (1)–(5) in the following form:

$$\partial_{t_1} d + \partial_{x_1} u_x + (\partial_{y_1} - At_1\partial_{x_1})u_y = 0, \tag{6}$$

$$\partial_{t_1} u_x + Au_y = -c_s^2 \partial_{x_1} d, \tag{7}$$

$$\partial_{t_1} u_y = -c_s^2(\partial_{y_1} - At_1\partial_{x_1})d + c_A^2[\partial_{x_1} b_y - (\partial_{y_1} - At_1\partial_{x_1})b_x], \tag{8}$$

$$\partial_{t_1} b_y = \partial_{x_1} u_y, \tag{9}$$

$$\partial_{x_1} b_x + (\partial_{y_1} - At_1\partial_{x_1})b_y = 0. \tag{10}$$

Let us perform the Fourier analyses of (6)–(10), expanding unknown functions with respect to *only* spatial variables x_1 and y_1,

$$F = \int dk_{x1} dk_{y1} \hat{F}(k_{x1}, k_{y1}, t_1) \exp[i(k_{x1}x_1 + k_{y1}y_1)], \tag{11}$$

where under F we imply physical quantities u_x, u_y, b_x, b_y, d. As a result we get

$$D^{(1)} = v_x + \beta(\tau)v_y, \tag{12}$$

$$v_x^{(1)} = -Rv_y - D, \tag{13}$$

$$v_y^{(1)} = -\beta(\tau)D + \sigma^2[1 + \beta(\tau)^2]b, \tag{14}$$

$$b^{(1)} = -v_y, \tag{15}$$

where hereafter $F^{(n)}$ will denote the nth order time derivative of F and $D \equiv i\hat{d}$, $b \equiv i\hat{b}_y$, $R \equiv A/(c_s k_{x_1})$, $\sigma^2 \equiv (c_A/c_s)^2$, $\tau \equiv c_s k_{x_1} t_1$, $\beta(\tau) \equiv k_{y_1}/k_{x_1} - R\tau \equiv \beta_0 - R\tau$, $v_x \equiv \hat{u}_x/c_s$, $v_y \equiv \hat{u}_y/c_s$.

Note that the wave number of a SFH along the flow shear $[k_y(\tau) \equiv k_{y_1} - Rk_{x_1}\tau]$ varies in time. This process of the linear drift of SFH in the $\mathbf{k}$ space will be referred to below as the linear drift. Note, also, that for small-scale perturbations in a *subsonic* flow $(V_0 < c_s)$ $R \ll 1$ [12].

The dimensionless total energy density of the perturbations in the $\mathbf{k}$ space we define as

$$E \equiv (|v_x|^2 + |v_y|^2)/2 + |D|^2/2 + \sigma^2(|b_x|^2 + |b_y|^2)/2. \tag{16}$$

If we introduce a new variable, $\psi \equiv D + \beta(\tau)b$, we can reduce the system (12)–(15) to the pair of ordinary differential equations of the second order:

$$\psi^{(2)} + \omega_1{}^2\psi + k(\tau)b = 0, \tag{17}$$

$$b^{(2)} + \omega_2{}^2 b + k(\tau)\psi = 0. \tag{18}$$

Further consideration becomes superfluous, since equations of this type are well known in the general theory of oscillations [3,22]. They describe coupled oscillations with two degrees of freedom. In particular, uncoupled eigenfrequencies appearing in (17) and (18) are $\omega_1 \equiv 1$ and $\omega_2(\tau) \equiv \sqrt{\sigma^2 + (1+\sigma^2)\beta(\tau)^2}$, while the coupling coefficient is $k(\tau) \equiv -\beta(\tau)$ [3,22]. The presence of shear in the flow $(R \neq 0)$ ensures temporal variability of one of the uncoupled eigenfrequencies $[\omega_2(\tau)]$ and the coupling coefficient $k(\tau)$. Note that a dependence of these quantities on time may be considered as adiabatic when $R \ll 1$ [12].

Fundamental frequencies of coupled (normal) oscillations are [3,22]

$$\Omega_{1,2}^2 = \frac{1}{2}[(\omega_1^2 + \omega_2^2) \mp \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4k^2}]. \tag{19}$$

In our case they correspond to slow and fast magnetosonic waves (SMW and FMW), respectively. Certainly, these frequencies also vary in time.

Since the oscillatory system, described by (17) and (18) has two degrees of freedom, its behavior may be determined by two functions, $\psi(\tau)$ and $b(\tau)$. Note that all physical quantities from (12)–(15) may be expressed through ψ, b and their derivatives: $D = \psi - \beta(\tau)b$, $v_x = \psi^{(1)} + Rb$, and $v_y = -b^{(1)}$.

The exact mechanical analogy of the oscillatory system, governed by the same kind of equations, is as follows. Let us

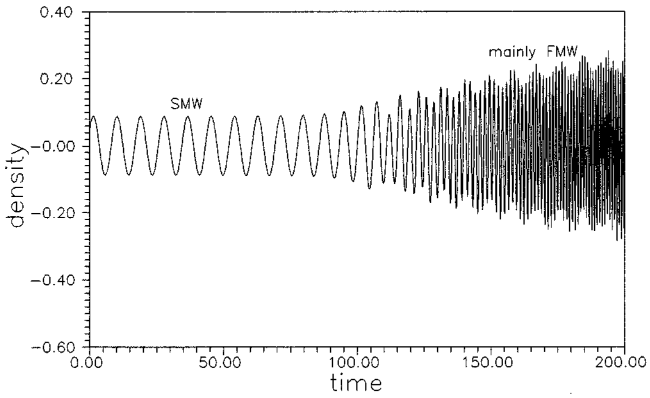

FIG. 1. Time dependence of $D(\tau)$ for $\sigma^2=1$, $\beta_0=5$, and $R=0.05$. The graph [numerical solution of Eqs. (12)–(15)] represents the transformation of SMW, with fundamental frequency $\Omega_1(\tau)$, into FMW with frequency $\Omega_2(\tau)$.

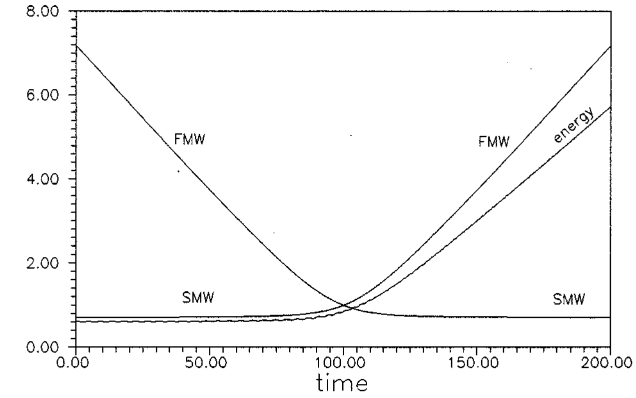

FIG. 2. Time dependence of the energy $[E(\tau)]$, SMW $[\Omega_1(\tau)]$, and FMW $[\Omega_2(\tau)]$ frequencies. Energy is normalized to reveal the connection between these quantities before, during, and after the transformation.

consider two coupled pendulums, the first one with constant uncoupled eigenfrequency $\omega_1=1$ and the second one whose uncoupled eigenfrequency $\omega_2(\tau)$ is slowly (adiabatically) varied by some external means (e.g., a variable length). The interpendulum coupling coefficient $k(\tau)$ is also time dependent. In [22], while considering a similar mechanical problem, the authors found two necessary conditions for the effectiveness of the energy exchange between the weakly coupled pendulums:

(1) there should exist a so-called ''degeneration region'' (DR) where $|\omega_1^2-\omega_2^2|\leqslant|k(\tau)|$ (in other words, in the case of weak coupling, this condition implies that $\omega_1\approx\omega_2$, which means that the maximum energy exchange between the pendulums occurs when they have approximately the same length);

(2) the DR should be passed slowly—in a time interval sufficiently exceeding the beating period $k(\tau)$: $|\omega_2^{(1)}(\tau)|\ll|k(\tau)|$.

Certainly these conditions should be valid for arbitrary oscillatory systems, governed by the same kind of differential equations. Thus, they can be straightforwardly applied in the analysis of an interaction between the modes with Ω_1 and Ω_2 (intermode or interwave coupling). Checking the applicability of these conditions in our problem, we can easily see that they are satisfied for a wide range of system parameters. The most suitable condition for the transformation of the MHD waves and corresponding energy transfer is when $\sigma^2\simeq1$ (i.e., $c_A\simeq c_s$ — equipartition between magnetic and thermal energies) and the transformation occurs near $\tau=\tau_*\equiv\beta_0/R$. In this case, condition (1) holds when $|\beta(\tau)|<1/2$, while condition (2) reduces to $\omega_2\gg2R$, which always holds when $R\ll1$. The actual course of the interaction process depends on the correlation between the width of the DR and the time interval [transformation time scale (TTS)] in which the system passes through DR. The value of the TTS depends on the R parameter—it increases with decreasing of R. Numerical simulations show that for certain values of R there occurs an almost complete transformation of SMW into FMW (if, initially, it was an excited SMW) and vice versa.

In Figs. 1 and 2 we demonstrate the simplest case—single transformation of SMW into FMW when $\sigma^2=1$, $\beta_0=5$, and $R=0.05$. In Fig. 1 we present the numerical solution for $D(\tau)$. The graph unambiguously shows the transformation of SMW, with fundamental frequency $\Omega_1(\tau)$, into FMW with frequency $\Omega_2(\tau)$. Figure 2 shows the graphs for $\Omega_1(\tau)$, $\Omega_2(\tau)$ and the curve for the perturbation energy. It is clearly seen that at $0<\tau<\tau_*$ the SMW energy remains almost constant, as it should, since the energy varies in the interval adiabatically $E(\tau)\sim\Omega_1$ and Ω_1 is almost constant there. For $\tau>\tau_*$, where the wave has already been transformed into FMW, $E(\tau)\sim\Omega_2$ and increases quasilinearly with the increasing τ. In particular, if initially we had a wave (SMW) that did not exchange an energy with the mean flow, after the transformation there appears a wave (FMW) that effectively extracts the shear energy from the regular flow. It is clear that this kind of transformation process may change radically the behavior of the flow.

In summary, we have shown in this simple example that transformation of linear MHD waves is possible in a homogeneous medium ($\rho_0=$const) when the regular velocity profile is inhomogeneous. It should be emphasized that this kind of transformation mechanism has been found in the framework of the nonmodal approach while it has not been perceived in the more traditional modal approach.

The nature of the wave transformation effect, discussed in this paper, qualitatively differs from the already known linear transformation mechanisms [5,6]. Density-inhomogeneity-induced mode transformation occurs permanently in a limited spatial area (across the density inhomogeneity), while in our case transformation of linear waves occurs in the whole volume, filled by the flow, in the limited time interval.

In general, realization of the transformation mechanism in shear flows takes place when the above specified conditions (1) and (2) are satisfied. It is also clear that for the existence of this phenomenon, the presence of at least two different wave modes is necessary. Evidently, this phenomenon should be realized in MHD and fusion plasma devices, where the existence of shear and numerous wave modes ensures the action of the mechanism for a wide range of system parameters. For the same reasons, such processes may also be important in terrestrial and astrophysical compressible hydrodynamic shear flows. In the latter case, the existence of several wave modes is ensured by differential rotation and pressure and gravity forces.

We believe that our analysis of Eqs. (17) and (18) enhances the mathematical theory of coupled oscillations. From this point of view, more detailed exploration of these equations may be valuable for general mathematical theory of coupled oscillations.

Finally, we would like to mention that the main goal of this paper was to outline the discovered transformation mechanism, while a detailed study of the phenomenon is in progress.

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